Errors of Measurement

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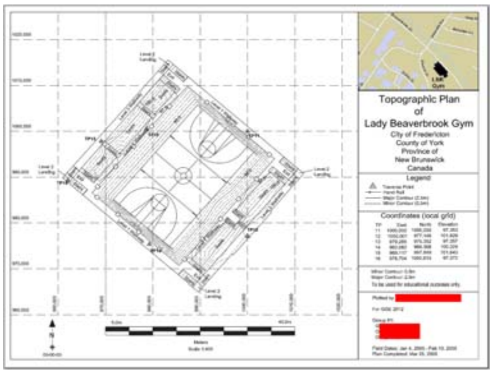
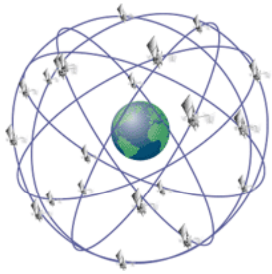




Objectives (Today)

- Random Errors
- Gaussian Distribution
- Reliability of Measurements
- Measures of Precision

Errors of Measurement

- **Surveying Errors**
  - *No measurement is exact*
  - *Every measurement contains errors*
  - *the true value of a measurement is "never" known*
  - *the exact error present is "always" unknown*

- **Sources of Errors: Instrument, environment, and operators**

  e.g.)

  -- instruments: imprecision and calibration

  -- environment: refraction of the earth, wind, temperature, etc.

  -- people: observation habit, judgement

- **Types of Errors: Gross, Random, and Systematic Errors**

Errors of Measurement

- **Gross Errors (blunders)**

  -- "not" errors, rather call them as mistakes

  -- eliminate by checking, re-measuring, and using right field procedures

- **Systematic Errors**

  -- consistent signs and magnitude, or obey certain rules

  -- calibrate instruments, applying corrections

- **Random Errors**

  -- have random sign and small in nature

  -- take more measurements, improve techniques

Errors of Measurement

- **Characteristics of Random Errors**

  -- small errors occur more often than larger ones
  -- positive and negative errors happen with the same frequency
  -- errors are normally within certain limit

- **Some Statistics of Random Errors**

  -- **most probable mean:** $\bar{x} = \frac{\Sigma_{i=1}^{n} x_i}{n}$

  -- **residuals:** $v_i = \bar{x} - x_i$

  -- **standard deviations:** $s = \sigma = \sqrt{\frac{\Sigma_{i=1}^{n} (\bar{x} - x_i)^2}{n-1}}$

  -- **standard error of the mean:** $\bar{s} = s / \sqrt{n}$

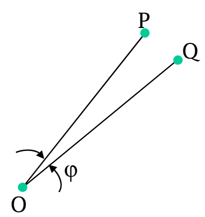
Errors of Measurement

- **Understanding the Meaning of the Residual Errors**

  -- For given observations: $L = (l_1, l_2, l_3, l_4, \cdots, l_n)$
  -- the arithmetic mean: $\bar{x} = \Sigma_{i=1}^{n} l_i / n$
  -- compute residuals: $v_i = \bar{x} - l_i$

  : residuals *represents* ***the degree of closeness*** of the repeated measurements of the same quantity to each other. Therefore, the residual values itself can be used in expressing the precision of $\bar{x}$

Ex1) Two observers A and B measured the same angle φ . Errors on each measurements are in the table shown.



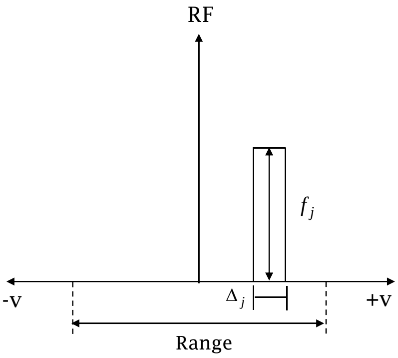
	A	B
φ1	+2”	+5”
φ2	-1”	-3”
φ3	+1”	+2”
φ4	0	-3”
φ5	-2”	-4”
φ6		+3”

Errors of Measurement

➢ Probability Distribution Histogram (PDH)

-- Assuming that all measurements are free of gross errors and corrected for all systematic errors.

1) calculate the estimated parameters: $\hat{x} = \bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$
2) calculate the residuals for each observations: $v_i = \hat{x} - x_i$
3) calculate the range of the residuals: $v_{\max} - v_{\min}$
4) divide the range into k equal intervals $\Delta_j (j = 1, 2, 3, \cdots, k)$
5) calculate the relative frequency for each interval $f_j = \frac{n_j}{n \cdot \Delta_j}$



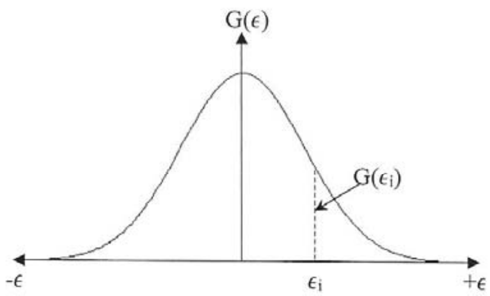
n_j : the number of residuals that fall within the boundaries of Δ_j

Errors of Measurement

➢ Probability Distribution Function (PDF)

-- Commonly accepted model by Gauss (Gauss or Normal PDF)

$$G(\varepsilon) = \frac{h}{\sqrt{\pi}} e^{-h^2 \varepsilon^2}, \quad e \approx 2.71828$$



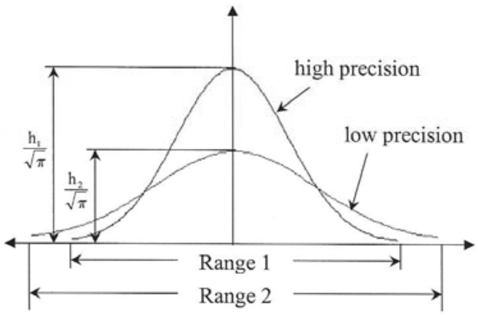
Johann Carl Friedrich Gauss (1777–1855), painted by Christian Albrecht Jensen

: h is the only parameter that completely describes the shape of the Gauss PDF

Errors of Measurement

-- when ε is zero: $G(0) = \frac{h}{\sqrt{\pi}}$

-- h is usually called "precision index". So, precision is proportional to h (amplitude)
-- since the area of the curve equals 1 (from the property of Gauss PDF, $\int_{-\infty}^{\infty} G(\varepsilon)d\varepsilon = 1$), so the higher the PDF (larger the h), the narrower the curve must become, i.e., less range (more precise)



Errors of Measurement

-- more general sense in the theory of errors of observations and least square adjustment, the distribution can be written like this (same as before):

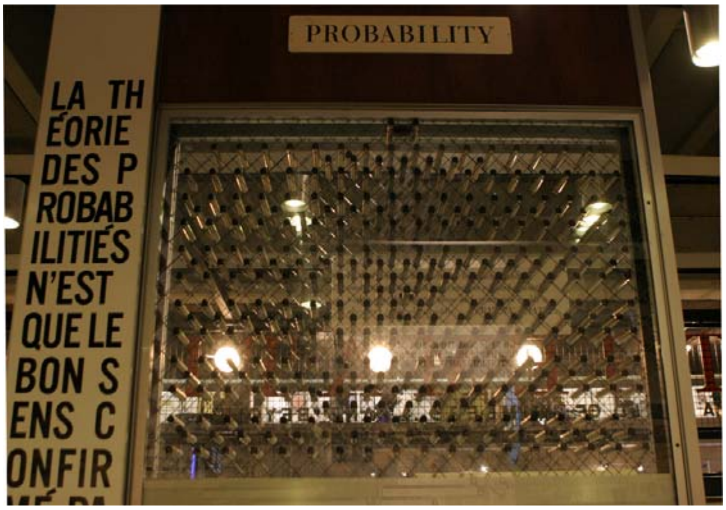
$$h = \frac{1}{\sigma_x \sqrt{\pi}}, \quad \varepsilon = x - \bar{x}$$

-- $\bar{x} = \mu$: mean of the distribution
-- σ_x^2 : variance of distribution
-- x : a continuous variable $-\infty \le x \le \infty$

$$f(x) = \frac{1}{\sigma_x \sqrt{2\pi}} e^{-\frac{(x-\bar{x})^2}{2\sigma_x^2}}$$ or $$X \sim N\left(\mu, \sigma_x^2\right)$$

Errors of Measurement

➢ **Bean Machine, Galton Box, ?????**



(from Boston Science Museum)

Errors of Measurement

(from Boston Science Museum)

Movie Clip

Errors of Measurement

EX2)

(Kavanagh, 2006)

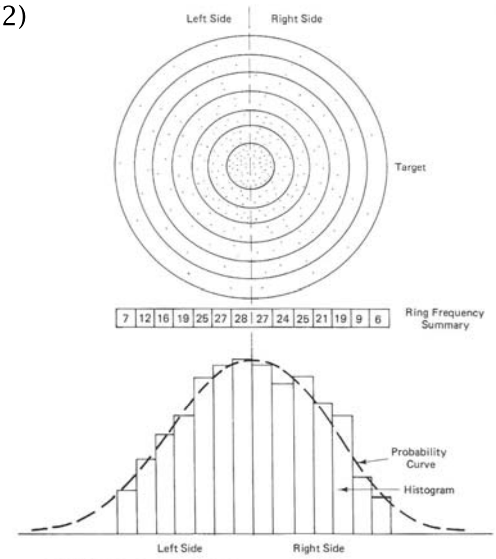


FIGURE A.1 Range target: 265 shots on target.

Table A.1 TARGET RING PROBABILITIES

Ring Number	Probability
1 (bull's-eye)	$\frac{28+27}{265} = 0.208$
2	$\frac{27+24}{265} = 0.192$
3	$\frac{25+25}{265} = 0.189$
4	$\frac{19+21}{265} = 0.151$
5	$\frac{16+19}{265} = 0.132$
6	$\frac{12+9}{265} = 0.079$
7	$\frac{7+6}{265} = 0.049$
	Total 1.000

Errors of Measurement

EX3)

(Kavanagh, 2006)

Table A.2 ANALYSIS OF RANDOM DISTANCE ERRORS ($v = x - \bar{x}$)

n	Distance x (m)	Residual, v	v^2
1	266.304	−0.0035	0.0000123
2	266.318	+0.0105	0.0001103
3	266.312	+0.0045	0.0000203
4	266.304	−0.0035	0.0000123
5	266.313	+0.0055	0.0000303
6	266.307	−0.0005	0.0000003
7	266.309	+0.0015	0.0000023
8	266.303	−0.0045	0.0000203
9	266.301	−0.0065	0.0000423
10	266.305	−0.0025	0.0000063
11	266.302	−0.0055	0.0000303
12	266.310	+0.0025	0.0000063
13	266.314	+0.0065	0.0000423
14	266.307	−0.0005	0.0000003
15	266.303	−0.0045	0.0000203
	Σx = 3994.612	Σv = −0.0005	Σv^2 = 0.0003565

1. Mean (most probable value)
: 3994.612/15 = 266.3075m

2. Standard error (SE)
: $\sqrt{\Sigma v^2/(n-1)} = \sqrt{0.0003565/14} = 0.0050m$

3. Standard error of the mean
: $SE/\sqrt{n} = 0.0050/\sqrt{15} = 0.0013m$

4. Probable Error (50 percent error)
: 0.6745*SE = 0.0034m

5. 90% of Error
: 1.6449*SE = 0.0082m

6. 95% of Error
: 1.9599*SE = 0.0098m

Errors of Measurement

➢ Reliability of Measurements

-- Recall that the true value (*t*) of certain unknown quantity can "*never*" be obtained. However, the estimate $\hat{x}$ can be determined. However, this $\hat{x}$ will be affected by the random errors. Therefore, we need a certain measure of the existing the random errors to describe its "goodness", "reliability", and "repeatibility".

-- This measure should help in "accepting" or "rejecting" certain observations depending on the desired precision. Two terms are commonly used to describe the reliability of measurements.

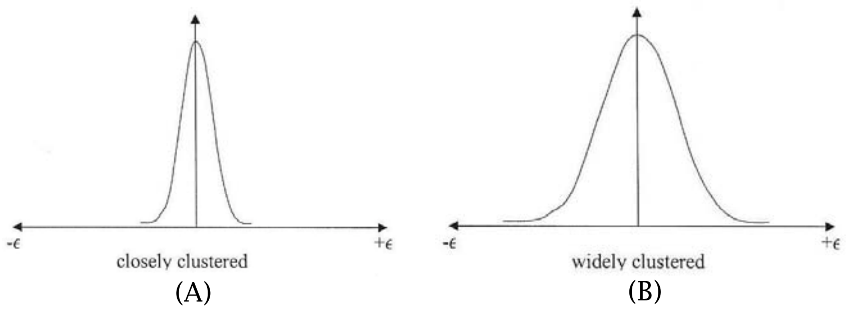
- Precision
- Accuracy

Errors of Measurement

ENGO361, Lecture Note in Calgary

➢ Precision

-- the degree of closeness of repeated measurements of the same quantity to each other. Precision is *affected only by random errors.*

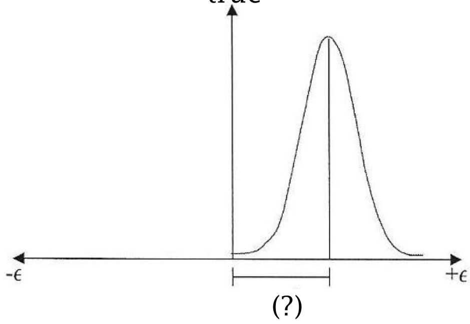


-- *We don't know the truth value in this case.*

Errors of Measurement

- **Accuracy**

-- the degree of closeness of repeated measurements to the true value. Accuracy is *affected by both random errors and systematic errors.*



-- *We know the truth value in this case.*

Errors of Measurement

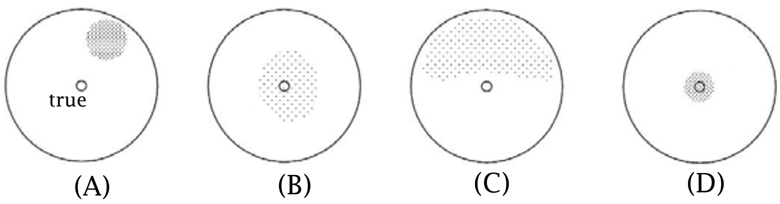
ENGO361, Lecture Note in Calgary

Ex) Precision and Accuracy

-- Precision: Internal Reliability

-- Accuracy: External Reliability

(in the absence of the systematic errors, both precision and accuracy are the same)



-- *How to quantify the precision?*

Errors of Measurement

➢ **Measure of Precision**

1) Average Error
2) Probable Error
3) Standard Deviation

✓ **Average Error**

: Arithmetic mean of the absolute value of errors

✓ **Probable Error**

: 50% of error (the limits under the curve is half of the total area). Those limits are $\pm 0.6745 \cdot \sigma$

Errors of Measurement

✓ **Standard Deviation (σ)**

: defined as the square root of the arithmetic mean of the sum of the square of the errors

: square of the standard deviation (σ^2) is known as the "variance" or "mean square error". So, σ is sometimes referred to as the root mean square error (RMSE)

Random Sample

t = unknown

$$\sigma^2 = \frac{1}{n-1} \Sigma_{i=1}^{n} v_i^2$$

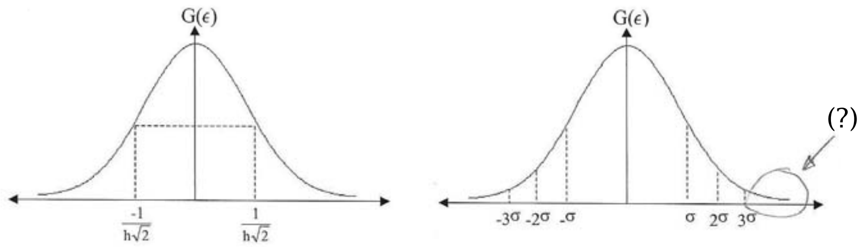
t = known (RMSE)

$$\sigma^2 = \frac{1}{n} \Sigma_{i=1}^{n} \varepsilon_i^2$$

t= true value, v_i = residuals, ε_i = true errors, n-1 = degree of freedom, or redundancy (redundancy=n-u, where n is number of observations, u is unknowns)

Errors of Measurement

: Unlike the precious two measures of precision, σ has the more distinguished properties.



: has geometric interpretation related to the PDF.

: also has a physical interpretation related to the PDF.

$$P(-\sigma \le \varepsilon \le +\sigma) = 0.683$$
$$P(-2\sigma \le \varepsilon \le +2\sigma) = 0.954$$
$$P(-3\sigma \le \varepsilon \le +3\sigma) = 0.997$$

} Confidence Interval

