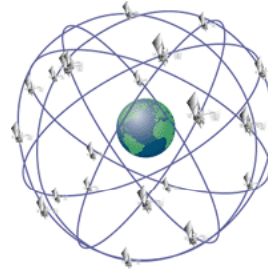
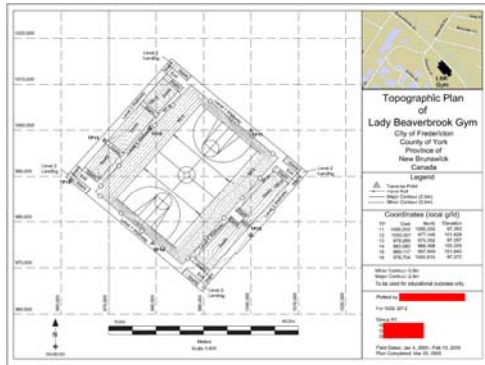


## Error Propagation

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1/00

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## Objectives (Today)

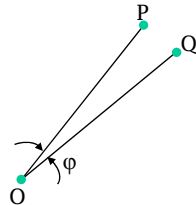
- ◆ Random Errors
- ◆ Gaussian Distribution
- ◆ Reliability of Measurements
- ◆ Measures of Precision
- ◆ **Weighting, Correlation, Covariance**
- ◆ **Error Propagation, Examples**

2/00

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## Review

**Ex1) Two observers A and B measured the same angle  $\varphi$  . Errors on each measurements are below.**



	A	B
$\varphi_1$	+2"	+5"
$\varphi_2$	-1"	-3"
$\varphi_3$	+1"	+2"
$\varphi_4$	0	-3"
$\varphi_5$	-2"	-4"
$\varphi_6$		+3"

Note: In the case that we are going to take "only one" observation, A is the best choice in terms of the precision. How is different if we "have to take all measurement" of both A and B?

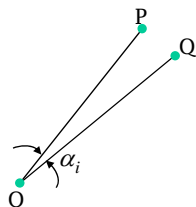
3/00

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## Weighting

### ◆ Direct Observations with Different Conditions

Ex2) Consider that two observers A and B are measuring **the same quantity** X (say the distance or angle) and that it is to estimate the best estimate of X by making the use of the two sets of observations



A	B
$\alpha_{A1}$	$\alpha_{B1}$
$\alpha_{A2}$	$\alpha_{B2}$
$\alpha_{A3}$	$\alpha_{B3}$
$\alpha_{A4}$	$\alpha_{B4}$
$\vdots$	$\vdots$
$\alpha_{An}$	$\alpha_{Bm}$

4/00

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## Weighting

### ◆ Direct Observations with Different Conditions

	Observer A	Observer B
Best Estimate	$\bar{\alpha}_A = \frac{\sum_{i=1}^n \alpha_{Ai}}{n}$	$\bar{\alpha}_B = \frac{\sum_{i=1}^m \alpha_{Bi}}{m}$
Precision	$\sigma_{\alpha_A} = \sqrt{\frac{\sum_{i=1}^n v_{Ai}^2}{n-1}}$ $v_{Ai} = \bar{\alpha}_A - \alpha_{Ai}$	$\sigma_{\alpha_B} = \sqrt{\frac{\sum_{i=1}^m v_{Bi}^2}{m-1}}$ $v_{Bi} = \bar{\alpha}_B - \alpha_{Bi}$

Note: Now, our original problem can be formulated as follows:

1) Given quantities:  $\bar{\alpha}_A, \sigma_{\alpha_A}$  and  $\bar{\alpha}_B, \sigma_{\alpha_B}$

$$\text{cf.) } \sigma_{\bar{\alpha}_A} = \sqrt{\frac{\sum_{i=1}^n v_{Ai}^2}{n(n-1)}}$$

2) Required: The best estimate of  $\alpha$  (angle)

5/00

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(ENGO361, Lecture Note)

## Weighting

### ◆ Direct Observations with Different Conditions

[Q] Could we say that  $\alpha = \frac{\bar{\alpha}_A + \bar{\alpha}_B}{2}$  ?

[A] No. Why?

: To account for the differences in accuracy between the two sets of observations, we need to introduce a new quantity called the weight:

$$P = \frac{1}{\sigma^2}$$

: A measurement with high precision will have small variance and vice versa.

$$\alpha = \frac{\sum P_i \alpha_i}{\sum P_i} = \frac{P_A \bar{\alpha}_A + P_B \bar{\alpha}_B}{P_A + P_B}$$

6/00

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## Weighting

### ◆ Properties

- P and n (number of repetition):

$$P_1 : P_2 = n_1 : n_2$$

- P and  $\sigma$  (errors):

$$P_1 : P_2 = \frac{1}{\sigma_1^2} : \frac{1}{\sigma_2^2}$$

### ◆ Ex) Most Probable Estimates

- When we don't count on the weights

$$L_0 = \frac{\sum l}{n} = \frac{l_1 + l_2 + l_3 + \dots + l_n}{n}$$

- When we count on the weights

$$L_0 = \frac{\sum (Pl)}{\sum P} = \frac{P_1 l_1 + P_2 l_2 + P_3 l_3 + \dots + P_n l_n}{P_1 + P_2 + P_3 + \dots + P_n}$$

7/00

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## Weighting

### ◆ Example 1)

We have two different groups, A and B, to determine the height difference between two points. The height determined by group A is  $50.446m \pm 0.009m$ , and by group B  $50.633m \pm 0.006m$ . What is the most probable estimates of the height for the two points?

Ans.)

8/00

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## Correlations

### ◆ Intro.

Normally, Geomatics (or surveying) problem include in the series of measurements with several unknown parameters.

The measurement can not usually be treated separately, but simultaneously. Both the effect of each quantity on the others and the ***statistical relationship between quantities*** must be taken into consideration in order to obtain solution of the unknowns.

A multivariate consists of several univariate, e.g.:

$$L = [a, b, c] , \text{ where } a, b, c \text{ are univariate.}$$

Therefore,

$$a = [a_1, a_2, \dots, a_n] \text{ with } \bar{a}, \sigma_a, \sigma_{\bar{a}}$$

$$b = [b_1, b_2, \dots, b_n] \text{ with } \bar{b}, \sigma_b, \sigma_{\bar{b}}$$

$$c = [c_1, c_2, \dots, c_n] \text{ with } \bar{c}, \sigma_c, \sigma_{\bar{c}}$$

9/00

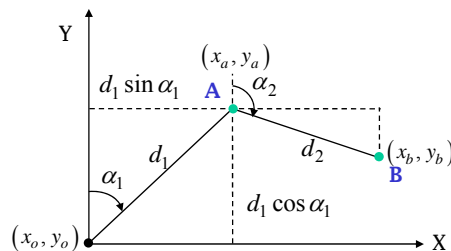
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## Correlations

### ◆ Example)

(ENGO361, Lecture Note)

Consider the following figure that needs to determine the coordinates, points A and B, based on the angular and distance measurement.



Unknowns:

$$x = [x_a, y_a, x_b, y_b]$$

Observations:

$$L = [\alpha_1, d_1, \alpha_2, d_2]$$

$$x_a = x_o + d_1 \sin \alpha_1, \quad x_b = x_o + d_1 \sin \alpha_1 + d_2 \sin \alpha_2$$

$$y_a = y_o + d_1 \cos \alpha_1, \quad y_b = y_o + d_1 \cos \alpha_1 + d_2 \cos \alpha_2$$

: Will any errors in  $\alpha_1$  and  $d_1$  affect the accuracy of  $(x_a, y_a)$  ?

: Will any errors in  $\alpha_1, d_1, \alpha_2$  and  $d_2$  will affect the accuracy of  $(x_b, y_b)$  ?

10/00

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## Correlations

### ◆ Covariance

Covariance is a ***measure of the degree of the correlation*** between any two components of a multivariate.

$$\sigma_{ab} = \frac{1}{(n-1)} \sum_{i=1}^n v_{a_i} v_{b_i} \quad \begin{array}{l} v_{a_i} = \bar{a} - a_i \\ v_{b_i} = \bar{b} - b_i \end{array}$$

Practically, all the variances and covariances of a multivariate are formulated into one matrix, called ***variance-covariance matrix*** (or simply covariance matrix)

$$C_L = \begin{bmatrix} \sigma_a^2 & \sigma_{ab} & \sigma_{ac} \\ \sigma_{ba} & \sigma_b^2 & \sigma_{bc} \\ \sigma_{ca} & \sigma_{cb} & \sigma_c^2 \end{bmatrix} \quad \begin{array}{l} : \text{symmetric} \\ : \text{diagonal is positive} \\ : \text{non-singular (=invertible)} \end{array}$$

11/00

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## Correlations

### ◆ Covariance

: Very special case, i.e., in the case of the ***statistically independent***. This means that there is no correlation between the parameter. The covariance matrix can then be like this:

$$C_L = \begin{bmatrix} \sigma_a^2 & 0 & 0 \\ 0 & \sigma_b^2 & 0 \\ 0 & 0 & \sigma_c^2 \end{bmatrix} = \text{diag}(\sigma_a^2 \ \sigma_b^2 \ \sigma_c^2)$$

: When you are dealing with a multivariate, the covariance matrix is very essential. This is because it ***provides weighting matrix***.

$$P = C_L^{-1}$$

Note: we had dealt with the weighting only with  $\sigma$ . If you know the variance-covariances, you have to use this stochastic process.

12/00

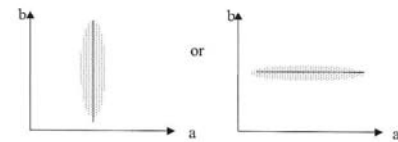
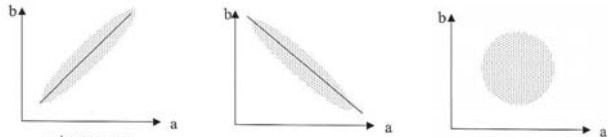
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# Correlations

## ◆ Correlation Coefficient

: a measure of how closely two quantities are related.

$$\rho_{ab} = \rho_{ba} = \frac{\sigma_{ab}}{\sigma_a \sigma_b} \quad : \text{Unitless and limit of } \pm 1$$



$0 \leq |\rho_{ab}| < 0.35$ , weak correlation  
 $0.35 \leq |\rho_{ab}| < 0.75$ , significant  
 $0.75 \leq |\rho_{ab}| < 1$ , strong correlation

(ENGO361, Lecture Note)

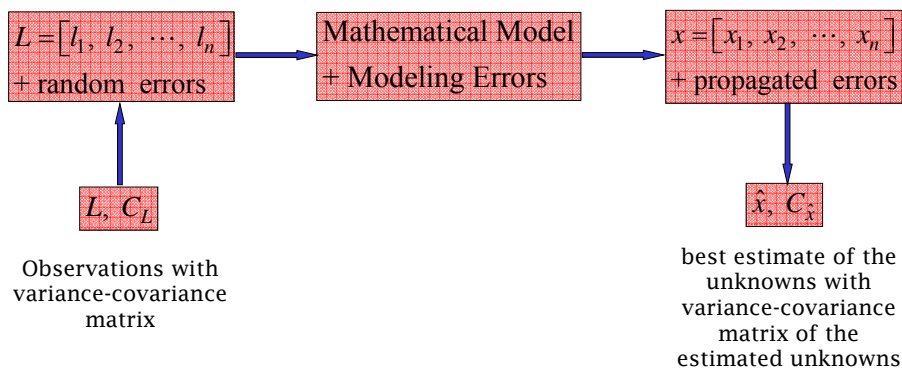
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# Error Propagation

## ◆ Error Propagation

: a process of evaluating the errors in estimated quantities (X) as functions of the errors in the measurement (L)



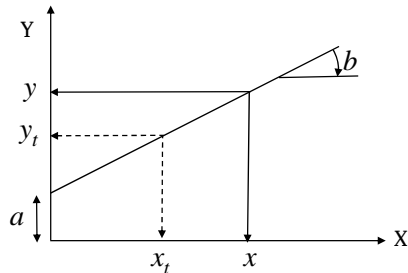
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## Error Propagation

### ◆ Concept

Suppose that a quantity ( $y$ ) is estimated from a measured quantity ( $x$ ) according to the following function:  $y = a + bx$  (1)



$$\text{true value: } y_t = a + bx_t \quad (2)$$

: defining the error of measurement as the measured value minus true, (1)-(2)

$$y - y_t = b(x - x_t)$$

$$dy = b \cdot dx \quad (3)$$

: Any error in a value,  $dx$  will introduce (i.e. propagate) an error of a value  $b \cdot dx$  in the  $y$  component

: Any error in estimating  $b$  (i.e., any error in math model) will introduce errors in  $y$

(ENGO361, Lecture Note)

15/00

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## Error Propagation

### ◆ Steps of Solution (Law of the Propagation of Variances)

1) construct the mathematical model (direct model):

$$x = f(l), \quad l = [l_1, l_2, \dots, l_n]$$

2) obtain the best estimate of  $x$ ,

$$\hat{x} = f(\hat{l})$$

3) Estimate the precision of  $\hat{x}$  based on error propagation law:

$$\sigma_{\hat{x}}^2 = \left( \frac{\partial f}{\partial l_1} \right)^2 \sigma_{\hat{l}_1}^2 + \left( \frac{\partial f}{\partial l_2} \right)^2 \sigma_{\hat{l}_2}^2 + \left( \frac{\partial f}{\partial l_3} \right)^2 \sigma_{\hat{l}_3}^2 + \dots + \left( \frac{\partial f}{\partial l_n} \right)^2 \sigma_{\hat{l}_n}^2$$

(Dare, 2007)

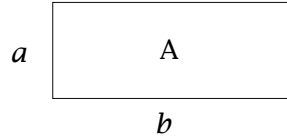
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## Error Propagation

◆ **Example 2.**

: Given  $\left\{ \begin{array}{l} x = A(\text{area}) \\ L = [a, b] \\ \bar{a} = 30m, \sigma_{\bar{a}} = 0.1m \\ \bar{b} = 40m, \sigma_{\bar{b}} = 0.2m \end{array} \right.$



Then, solve for  $\hat{A}$  and  $\sigma_{\hat{A}}$

1) mathematical model  $x = f(l) \rightarrow A = a \cdot b$

2) best estimate  $\hat{x} = f(\hat{l}) \rightarrow \hat{A} = \hat{a} \cdot \hat{b} = 30m \cdot 40m = 1200m^2$

3) Estimate the precision of  $\hat{x} \rightarrow \sigma_{\hat{A}}$  based on error propagation law:

$$\sigma_{\hat{A}}^2 = \left(\frac{\partial A}{\partial a}\right)^2 \sigma_a^2 + \left(\frac{\partial A}{\partial b}\right)^2 \sigma_b^2 = b^2 \sigma_a^2 + a^2 \sigma_b^2 = (40m)^2 (0.1m)^2 + (30m)^2 (0.2m)^2$$

$$= 1600 \cdot 0.01m^2 + 900 \cdot 0.04m^2 = 52m^4, \quad \sigma_{\hat{A}} = 7.211m^2, \quad \therefore A = (1200 \pm 7.211)m^2$$

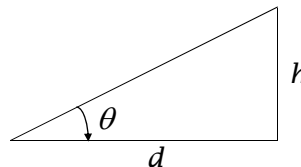
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## Error Propagation

◆ **Example 3.**

: Given  $\left\{ \begin{array}{llll} \text{mean} & \sigma & n & \\ d & 56.78m & 2cm & 4 \\ \theta & 9^\circ 12' 7'' & 30'' & 9 \end{array} \right.$



Then, solve for  $\hat{h}$  and  $\sigma_{\hat{h}}$

1) mathematical model  $x = f(l) \rightarrow h = d \tan \theta$

2) best estimate  $\hat{h} = \bar{d} \tan \bar{\theta} = 56.78m \cdot \tan 9^\circ 12' 7'' = 9.1983m$

3) Estimate the precision of  $\hat{x} \rightarrow \sigma_{\hat{h}}$  based on error propagation law:

$$\sigma_{\hat{h}}^2 = \left(\frac{\partial h}{\partial d}\right)^2 \sigma_d^2 + \left(\frac{\partial h}{\partial \theta}\right)^2 \sigma_{\theta}^2, \quad \sigma_{\bar{d}} = \frac{\sigma_d}{\sqrt{4}} = \frac{2}{\sqrt{4}} = 1cm, \quad \sigma_{\bar{\theta}} = \frac{\sigma_{\theta}}{\sqrt{9}} = \frac{30''}{\sqrt{9}} = 10''$$

$$\frac{\partial h}{\partial d} = \tan \theta, \quad \frac{\partial h}{\partial \theta} = d \sec^2 \theta, \quad \therefore \hat{h} = 9.1983 \pm \sigma_{\hat{h}} = ?, \quad (\text{cf.}) 1\text{rad} = 206264'' .806$$

18/00

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